

A PROGRAMMING MODEL
FOR CORPORATE FINANCIAL MANAGEMENT

by

Stewart C. Myers and Gerald A. Pogue

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I. INTRODUCTION

The theory of financial management now includes detailed considerations of investment and financing decisions, of dividend policy, and of most other aspects of corporate finance. But there is a clear tendency to isolate these decisions in order to analyze them. To take a simple example, consider the capital budgeting rules which depend on an exogenous "cost of capital." These must assume that the firm's financing decision is taken as given or is independent of the investment decision, even though neither theory or practical considerations support such a separation.

Financial management really requires simultaneous consideration of the investment, financing, and dividend options facing the firm. The purpose of this paper is to present a mixed-integer linear programming model which includes these decisions and their interactions. The model is based on recent advances in capital market theory, but at the same time it recognizes certain additional considerations that are manifestly important to the financial manager. Thus, we believe that the model has both theoretical interest and the potential for practical application.

Before plunging into detail, it may be helpful to state the main features of our approach in very general terms.

The linear programming model follows from two propositions of modern capital market theory, namely:

- (1) That the risk characteristics of a capital investment opportunity can be evaluated independently of the risk characteristics of the firm's existing assets or other opportunities.
- (2) The Modigliani-Miller result that the total market value of the firm is equal to its unlevered value plus the present value of taxes saved due to debt financing.

Thus, the firm is assumed to choose that combination of investment and financing options that maximizes the total market value of the firm, specified according to the two axioms. The major constraints are a debt limit (specified as a function of the value and risk characteristics of the firm's assets and new investment) and a requirement that planned sources and uses of funds are equal. In addition, there are constraints on liquidity, dividend policy, investment choices (due to mutually exclusive or contingent options), etc. We do not argue that every one of these additional constraints will be relevant in a practical situation, rather we have included them for the sake of completeness.

The model has several important features:

- (1) The objective is specified in terms of the firm's market value, not in terms of someone's utility function.
- (2) There is no restriction on the risk characteristics of the investment opportunities considered.
- (3) The model does not rely on the weighted average cost of capital, and thus avoids the difficulties associated with that concept.¹

(4) The model is linear in the decision variables, thus avoiding the computational difficulties associated with non-linear programming.² It requires mixed integer programming for solution, but practical size problems can be efficiently handled with existing computer codes.

Of course no model for overall financial management can or should be entirely original -- our contribution is the combination of ingredients, not the ingredients themselves. Our debt to many other authors will be evident as we present the model piece by piece.³

II. OBJECTIVE FUNCTION

The objective is to maximize the current market value of the net worth of the firm, and thereby to maximize the wealth of current shareholders. The market value of net worth at the beginning of period t will be written as NW_t , and the objective is thus to maximize NW_1 .

The firm is, of course, faced with certain immediate decisions. But in general NW_1 will also depend on future financing and investment opportunities and the firm's future financial strategy. Thus, the problem is to find the optimal financial plan for the present ($t = 1$) and for future periods out to a horizon period ($t = 2, 3, \dots, H$).

Perfect capital markets will be assumed except for the imperfections specifically included in the model.

1. Decision Variables

We assume that management has identified a series of investment opportunities, or projects, including both current and future opportunities. The decision variables are:

$$X_k = \begin{cases} 1, & \text{if project } k \text{ is accepted} \\ 0, & \text{if project } k \text{ is rejected.} \end{cases}$$

The firm also has the opportunity to place funds in liquid assets (marketable securities):

$$L_t = \text{dollar investment in liquid assets at the start of period } t$$

The major financing decisions are the amount and type of debt issued, the amount of equity issued, and total cash dividends paid. For the case of debt,

$$Y_{ti} = \text{dollar value of debt issued at the start of } t, \text{ in risk class } i.$$

The second subscript on Y is needed because the risk of debt (to the lender) depends on the firm's total debt obligation relative to its debt capacity. Thus, the interest rate charged the firm will be a function of the total debt issued. This is handled in the model by assuming that the firm can issue a certain amount of prime debt ($i = 1$), but that to issue more the firm must switch to medium grade ($i = 2$), and so on. For analytical convenience it is assumed that debt in each risk class is subordinate to debt of higher grades.⁴

We could add a third subscript to distinguish debt options by their type and maturity. Clearly, management has to distinguish 5-year from 20-year loans; but it might also be that a 5-year loan from a bank has, say, a different repayment schedule from a 5-year note issued via an underwriter. In order to simplify the presentation of the model we ignore these considerations in this section of the paper, and assume that the firm is concerned only with the total stock of debt outstanding in t , not its composition.

The decision variables for dividend and equity issues are

E_t = amount of equity issued at the start of t

D_t = dollar value of dividends paid at t .

There are a number of other variables in the objective functions -- various slack variables, for example, and also variables introduced to reflect the implicit costs of violating certain constraints. For example, management may wish to assign a cost to deviations from a target dividend payment. However, we will not discuss these variables further at this point since their rationale depends on the model's constraints.

2. The Status Quo

Usually the firm will have existing assets and liabilities taken as given for purposes of analysis. This is included in the model by specifying certain "autonomous" variables which are forced into the solution. The existing assets are treated as the 0th project, with $X_0 = 1$. The existing debt levels in various risk classes are given by Y_{0i} , where Y_{0i} is constrained equal to the actual debt outstanding in class i .

Similarly, D_0 and L_0 represent existing levels of dividend payments and liquid assets respectively.

3. Form of the Objective Function

In the most general sense the objective is to maximize NW_1 , which is a function of the X_k 's, L_t 's, Y_{ti} 's, D_t 's, and E_t 's. The problem is to specify NW_1 as a linear function of these variables.

Investment Decisions

We will begin with the "base case" of (1) all-equity financing, (2) no liquid assets, and (3) irrelevance of dividend policy. The third condition is defined in the sense of the well-known Miller-Modigliani proof [11], which in the present context implies that NW_1 is independent of the D_t 's and E_t 's. In this case we propose that

$$NW_1 = \sum_{k=1}^N X_k A_k^1 \quad (2-1)$$

where A_k^1 is the net present value of project k evaluated at the start of $t = 1$.

This specification rests on two assumptions. The first is "risk independence," i.e., that risk characteristics of capital investment

opportunities can be evaluated independently of the risk characteristics of the firm's existing assets or other opportunities. This means that the market value of a portfolio of projects is equal to the sum of the market values of the individual projects (assuming the projects could be split off from the firm and separately financed). It also means that diversification is more efficiently and flexibly accomplished within investors' portfolios than within the firm.

There are several proofs that risk independence is a necessary condition for equilibrium in perfect security markets.⁵

The second assumption underlying Eq. (2-1) is the absence of causal or "physical" interdependencies between project cash flows. Under this assumption we must rule out competitive or complementary projects. Obviously, if adoption of project j reduced the cash returns of project k , then A_k^1 would depend on X_j and Eq. (2-1) would not hold. However, causal interactions can be treated by introducing dummy projects whose acceptance is contingent on adoption of the interacting projects. This is discussed in detail later in the paper.

The important thing about Eq. (2-1), besides its linearity, is that it makes no special assumptions about the risk characteristics of investment projects. The manager computes A_k^1 taking project k 's risk into account and then runs the model.

Of course, the model does not specify how the value of each investment opportunity is to be computed. For practical purposes, we propose

$$A_k^1 = \sum_{t=1}^{\infty} \frac{C_k(t)}{(1 + \rho_k)^{t-1}}, \quad (2-2)$$

where: ρ_k = the cost of capital for project k, assuming the base case of all-equity financing, dividend irrelevance, etc.

ρ_k is not a weighted average cost of capital.

$C_k(t)$ = the expected after-tax cash flow from project k in t.

$C_k(t) < 0$ indicates a net cash outflow (investment). All cash flows are assumed to occur at the beginning of the period.

Debt Financing

Now we add the debt options:

$$NW_1 = \sum_{k=0}^N X_k A_k^1 + \sum_{\tau=0}^H \sum_{i=1}^I Y_{\tau i} F_{\tau i}^1 \quad (2-3)$$

where $F_{\tau i}^1$ is the present value per dollar of class i debt issued in τ , computed at the start of period 1. The variables for $\tau = 0$ refer to the firm's existing debt.

Since the investment options are evaluated in terms of the base case of all-equity financing, we must interpret $\sum_{\tau} \sum_i Y_{\tau i} F_{\tau i}^1$ as the increase in current net worth due to a planned shift from all-equity financing to a mixed capital structure. Calculation of the $F_{\tau i}^1$'s must therefore be based on some theory of optimal capital structure. Although the model does not rest on any particular theory (except for the assumption of linearity) the Modigliani-Miller (MM) theory is the natural choice since we have assumed perfect capital markets. MM state that the total value of the firm is equal to its unlevered value plus the present value of tax savings due to debt financing.⁶

This idea can be implemented in the present context by a simple discounted present value calculation. To simplify exposition, we will assume that the borrowing rates ρ_i are expected to be constant during the firm's planning horizon. In this section we are not concerned with the maturity of debt issued, and we can analyze debt issued in t as if it had to be repaid or refinanced in $t + 1$.⁷ Thus Y_{ti} is interpreted as the total stock of "new" debt in class i that is outstanding from t to $t + 1$. (There may also be "old," that is autonomous, debt still outstanding, denoted by Y_{0i} .) Then we can write down the after-tax cash flows and discount these flows back to period 1 at ρ_i , the prevailing interest rate for class i debt.

Thus,⁸

$$F_{ti}^1 = \frac{f_{ti}(t)}{(1+\rho_i)^{t-1}} + \frac{f_{ti}(t+1)}{(1+\rho_i)^t} \quad (2-4)$$

where $f_{ti}(t)$ is the general notation for the after-tax cash flows occurring in t per dollar of class i debt issued in t .

Equation (2-4) looks more complicated than it really is. The immediate effect of issuing \$1 of debt is a \$1 cash inflow. Thus $f_{ti}(t) = 1$. But there is a cash outflow the next period which is to interest plus principal repayment less the tax shield on interest. Thus,

$$f_{ti}(t+1) = -(1 + \rho_i) + T_c \rho_i$$

where T_c is the marginal corporate tax rate. Discounting back to the start of period 1,

$$F_{ti}^1 = \frac{T_C \rho_i}{(1 + \rho_i)^t} \quad (2-5)$$

This is precisely the present value of tax savings due to \$1 of class 1 debt issue in t.

Eq. (2-5) rests on the assumption that the firm borrows at the prevailing interest rate ρ_i , and the market capitalizes the tax shield $T_C \rho_i$ at the same rate. This is reasonable since the firm can realize the tax shield only if the firm can pay the interest. Thus the tax shields and interest payments have similar risk characteristics.⁹

Bankruptcy Costs

At this point we must recognize an issue left unclear in the MM propositions. If the choice between debt and equity is irrelevant except for tax savings, then why not finance with 99.99 percent debt? In our opinion there is no simple response, but we can recognize three limits that are practically important, namely (1) capital rationing,¹⁰ (2) managerial risk aversion and (3) bankruptcy costs. The first two items would be reflected in debt capacity constraints (discussed below) and do not affect the objective function. Bankruptcy costs, however, do enter the objective function.

We interpret "bankruptcy costs" broadly, including not only the costs of bankruptcy and reorganizations, but also any otherwise suboptimal actions taken by the firm in order to avoid bankruptcy. The costs are related to planned debt commitments in the various debt classes.

For example, assume the firm plans to have \$1 of debt outstanding in class 2, period 1. (Class 1 = 1 debt is assumed to be prime and thus leads to low and perhaps negligible bankruptcy costs.) We impute a cost of, say, \$0.02.

to this liability. This amount is the decline in the market value of the firm due to the possibility of bankruptcy due to the debt commitment.¹¹

The general notation for these costs is $p_i^b(\tau)$, the present value at the beginning of the planning horizon ($t = 1$) of bankruptcy costs associated with \$1 of class i debt outstanding in period τ .

The penalty costs increase with i , since a move to a higher risk class of debt by definition implies a greater chance of actual or near-bankruptcy.

Also, the period 1 present value will decline as τ increases, according to

$$p_i^b(\tau) = \frac{p_i^b(1)}{(1 + g^b)^{\tau-1}} \quad (2-6)$$

where g^b is a positive discount rate.

The actual values of the $p_i^b(\tau)$'s will probably be estimated subjectively which is dangerous in that the model user may employ them as fudge factors. We accept this danger in view of their practical importance.

Liquid Assets

We can now extend the model to include liquid assets. For simplicity of presentation we assume liquid assets to be one-period riskless securities yielding a rate of return ρ_L .

Consider a \$1 investment at the beginning of period t which returns \$1 plus $\rho_L(1 - T_C)$ after corporate taxes at the beginning of period $t + 1$. The present value at the start of period 1 of this investment, A_{tL}^1 , is given by

$$A_{tL}^1 = \frac{-1.0}{(1+\rho_L)^{t-1}} + \frac{1.0 + \rho_L(1-T_C)}{(1+\rho_L)^t} = \frac{-T_C\rho_L}{(1+\rho_L)^t} \quad (2-7)$$

Note that A_{tL}^1 will be negative for reasonable values of ρ_L and T_C . The reason is that liquid assets are equivalent to negative debt: if $\rho_L = \rho_1$, then $A_{tL}^1 = -F_{t1}^1$, as can be seen by comparing Eqs. (2-5) and (2-7). The liquid assets create an additional tax rather than a tax shield. The fact that debt and liquid assets have mutually offsetting effects on NW means that ρ_L must be a bit less than ρ_1 to insure that the model has a determinant solution. However, this makes sense since the liquid assets are considered riskless; whereas the first class of debt, while "prime," is not absolutely risk free.

Dividend Policy and Equity Issues

If we now add dividend and equity issue decisions to the factors already discussed, we have

$$\begin{aligned}
 NW_1 = & \sum_{k=0}^N X_k A_k^1 + \sum_{\tau=0}^H \sum_{i=1}^I Y_{\tau i} [F_{\tau i}^1 + P_i^b(\tau)] \\
 & + \sum_{\tau=0}^H L_{\tau} A_{\tau L}^1 + \sum_{\tau=1}^H P^e(\tau) E_{\tau} + \sum_{\tau=1}^H P^d(\tau) D_{\tau},
 \end{aligned} \tag{2-8}$$

where the last two terms are the effects of planned equity issues and dividend policy, respectively, on the market value of net worth.

The model does not rely on any specific theory of optimal dividend policy, except for the assumption of linearity. However, the natural starting point is again the MM theory. According to MM's proof,¹² dividend policy is irrelevant (i.e., it does not affect shareholders' wealth) assuming investment policy is given. Any change in dividend payment can be offset by a change in the amount of equity issued or

shares repurchased, and the change in cash dividend received by the firm's shareholders is exactly offset by a capital gain or loss due to the issue or retirement of shares. In our model, this implies $p^e(\tau) = p^d(\tau) = 0$. That is, given values for the X_k 's, the planned values of E_t and D_t have no effect on NW_1 .

However, there is at least one market imperfection that cannot be ignored: the transaction costs associated with new equity issues. Thus, we interpret $p^e(\tau)$ as the cost per dollar of equity issued in period τ ,¹³ with

$$p^e(\tau) = \frac{p^e(1)}{(1+g^e)^{\tau-1}} \quad (2-9)$$

The discount rate g^e reflects the fact that issue costs are better delayed, simply because of the time value of money. Another complication is that total issue costs are not a strict linear function of the amount of equity issued. There is a substantial fixed cost associated with any issue, for example. Thus, we must turn to a piecewise linear function with $F^e(\tau)$ representing the fixed cost and $v^e(\tau)$ the variable cost per dollar of equity issued. We also require an equity issue decision variable, Z_t^e , which takes the value zero if no equity is issued in period t and one otherwise. This is discussed more fully in the next section.

Although we know that equity issues are undesirable, it is less clear how dividend increases affect NW_1 . Some would make $p^d(\tau)$ positive because of market imperfections or irrationality. We are more inclined to the view that $p^d(\tau)$ is negative due to unfavorable tax treatment of cash dividends relative to capital gains income. That is, we interpret $p^d(\tau)$ as the extra taxes paid by the firm's shareholders when they received a dollar of cash income in τ instead of a dollar of capital gains. However, it is difficult to prove either view empirically.¹⁴

Informational Content of Dividends and Reported Earnings

There is one more consideration which enters into the objective function. So far we have assumed that investors have access to the same information about the firm's plans and prospects as do the firm's managers. This is often not true; consequently, we must interpret NW_1 as the intrinsic value of the firm's net worth, and recognize that the actual market value may not equal NW_1 when information is limited.

This sort of market imperfection makes it hard to specify the firm's objective with absolute clarity. Nevertheless, it seems more reasonable and natural to maximize intrinsic value than any evident alternative. In fact, the large corporation -- providing limited liability and a means for delegation of financial decision making authority -- seems a natural response, given the task of maximizing value in the face of informational limitations.

The major difficulty arises when pursuit of intrinsic value leads to systematically false signals. Suppose the firm undertakes a large and attractive investment. It cuts the dividend payment to finance the investment. The probable result is a temporary drop in stock price, since the dividend cut will be interpreted as a sign of management pessimism about future earnings. The price will stay at a depressed level until the true situation comes to light, and in the meantime shareholders may sell at unfairly low prices.

A related problem is that most large investments generate relatively little reported income during an initial "start-up" period (which may last several years). This does not necessarily indicate a poor investment decision, but rather a bias in the calculation of accounting

income. Thus, adoption of a project can generate a false signal concerning the firm's long-run earnings.

There are basically two ways of reacting to this sort of problem. One is to tell the market the true cause for the low earnings or dividend decrease. The other is to rearrange the firm's financial decisions so as to maintain dividends and reported earnings. The choice between these two strategies depends on the feasibility of the former versus the cost of the latter.

In our model there are two sets of constraints which allow management to "cost out" the policies of smoothing dividends and reported earnings. (This follows Chambers [3a] and Lerner and Rapoport [8].) The constraints establish target dividend and reported earnings growth rates, g_d and g_r and impose penalties $p_j^{dc}(\tau)$ and $p_j^{re}(\tau)$ if the target levels are not met.

$p_j^{dc}(\tau)$ is the present value at the start of period 1 of penalty costs associated with \$1 of dividend cut in period τ . The penalty depends on the amount of the cut in steps indicated by $j = 1, \dots, P$.

Similarly, $p_j^{re}(\tau)$ is the cost as of period 1 per dollar penalty class j reported earnings reduction in period τ .

Summary

Now we can put all of these pieces together. The objective function is to maximize the intrinsic value of W_1 , the firm's net worth, evaluated at the start of $t = 1$. NW_1 is the sum of

- (1) The present value of the firm's investments evaluated assuming perfect markets and all-equity financing.

- (2) Plus the net present value of investments in liquid assets.
- (3) Plus the present value of debt financing versus all-equity financing, due to tax savings, net of bankruptcy costs.
- (4) Minus the ^{present value of transaction} costs of planned equity issues (fixed plus variable costs).
- (5) Minus the ^{present value of} tax penalties associated with dividend payments.
- (6) Minus penalties assessed when target growth rates for dividends and reported earnings are not met.

The next section presents a detailed discussion of the decision variables and constraints associated with the various net worth penalty costs. Following this we present a complete algebraic expression for the objective function.

III. THE CONSTRAINTS

As discussed above, the goal of financial planning is to select a financial plan which will maximize the value of the shareholder's equity. The financial plan consists of a set of investment and financing decisions covering the planning horizon of the firm. In this section we define the restrictions on the set of feasible plans. These constraints can be grouped into six categories, as follows.

- (1) Cash flow constraints, insuring that planned sources and uses of funds are equal.
- (2) Debt capacity: these constraints limit the amount of debt the firm can issue in various risk classes.
- (3) Liquidity reserve: these constraints ensure that sufficient "slack" exists in the financial plan to provide protection against the uncertainties associated with projected cash requirements. The "slack" takes the form of a liquidity reserve, composed of unused borrowing capacity plus the liquid assets held by the firm.
- (4) Investment restrictions: these constraints are necessary to allow physical dependencies among projects.
- (5) Equity issue costs: these constraints are necessary to represent the costs associated with new equity issues.
- (6) Information effects: these constraints represent the costs associated with "erratic" dividends and reported earnings.

We now proceed to a detailed discussion of each of the constraint types. To assist the reader, a summary of notation is given in Table 3-1.

1. Expected Sources and Uses Constraints

For every feasible financial plan, expected cash requirements in each period must be exactly matched by sources of funds. Thus,

Table 3-1

Summary of Major Variables Used

1. Decision Variables

X_k	Project zero-one decision variable, $k = 0, 1, \dots, N$.
L_t	Stock of liquid assets in period t , $t = 0, \dots, H$.
$Y_{\tau i}$	Amount of risk class i debt issued in period τ . $\tau = 0, 1, \dots, H; \quad i = 1, \dots, I$.
E_t	Amount of equity issue in period t .
Z_t^e	Equity zero-one decision variable.
D_t	Aggregate dividends paid in period t .
$\Delta_{\tau j}^d$	Amount of penalty class j dividend reduction in period τ .
$\Delta_{\tau j}^r$	Amount of penalty class j reported earnings reduction in period τ .

2. Parameters: Investment Options

A_k^t	Present value at t of project k .
σ_{Ak}^t	Standard deviation of \tilde{A}_k^t ; $\sigma_{Ak}^1 = 0$, however.
δ_{kt}	Correlation coefficient between \tilde{A}_k^t and \tilde{A}_0^t .
$C_k(t)$	Period t after-tax cash flows.
$C_{ck}(t)$	Cumulative cash flows to period t , i.e., $\sum_{t=1}^t C_k(t)$

Table 3-1 (Continued)

$\sigma_{ck}(t)$	Standard deviation of $\tilde{C}_{ck}(t)$.
γ_{kt}	Correlation coefficient between $\tilde{C}_{ck}(t)$ and $\tilde{C}_{c0}(t)$.
ρ_k	Discount rate for project k, assuming all- equity financing.
RE_k^t	Contribution of project k to reported earnings in year t.
A_{TL}^t	Present value at t per dollar of L_T .
ρ_L	Rate of return on liquid assets, before tax.

3. Parameters: Financing Options

F_{Ti}^t	Present value at t per dollar of class i debt <u>issued</u> in τ .
$f_{Ti}(t)$	After-tax period t cash flow per dollar of class i debt issued in τ .
β_{Ti}^t	Dollars outstanding in t per dollar of class i debt issued in τ .
S_{it}	Stock of class i debt in period t.
Z_{it}	Maximum amount of class 1 through i debt in period t.
ρ_i	Interest rate on class i debt.

For every firm in the sample

Table 3-1 (Continued)

4. Parameters: Net Worth

- \tilde{NW}_t Net worth at t .
- \tilde{TA}_t Total assets at t .
- $\sigma(\tilde{NW}_t)$ Standard deviation of \tilde{NW}_t .
- $\sigma(\tilde{TA}_t)$ Standard deviation of \tilde{TA}_t .

5. Parameters: Penalty Costs

All penalty costs are present value as of the start of period t .

- $p_{ti}^b(\tau)$ Bankruptcy costs per dollar of risk class i debt outstanding in period τ .
- $v_t^e(\tau)$ Variable issue costs per dollar of equity issued in period τ .
- $F_t^e(\tau)$ Fixed cost of equity issued in period τ .
- $p_t^d(\tau)$ Tax penalty per dollar of dividend payments in period τ .
- $p_{tj}^{dc}(\tau)$ Penalty cost per dollar of period τ dividend reduction (penalty class j).
- $p_{tj}^{re}(\tau)$ Penalty cost per dollar of period τ reported earnings reduction (penalty class j).
- g_d Target dividend growth rate.
- g_r Target reported earning growth rate.

expected sources and uses of cash must net to zero in each decision period. That is,

$$\begin{array}{rcl}
 \text{After tax cash} & & \text{After tax cash} & & \text{Net proceeds} \\
 \text{flow from} & + & \text{flow from debt} & + & \text{of equity} \\
 \text{projects} & & \text{options} & & \text{issued} \\
 \\
 - \text{Dividends} & - & \text{Increase in} & = & 0 \\
 \text{paid} & & \text{liquid assets} & &
 \end{array}$$

Stated in terms of the specific decision variables, this relationship becomes, for each period $t = 1, \dots, H$,

$$\sum_{k=0}^N X_k \cdot C_k(t) + \sum_{\tau=0}^t \sum_{i=1}^I Y_{\tau i} \cdot f_{\tau i}(t) \quad (3-1)$$

$$+E_t[1 - v_t^e(t)] - Z_t^e F_t(t) - D_t - [L_t - (1 + (1-T_c)\rho_L) L_{t-1}] = 0$$

Note that the cash flows from the firm's existing projects and debt financing are included in the equation as X_0 and Y_{01} .

2. Debt Capacity Constraints

The amount and quality of the debt a firm can issue obviously depends on the nature of its assets. The more claims the firm issues against its assets, the lower the quality (rating) of these claims and the higher the rate of interest demanded by lenders.

We assume that the firm can issue claims in I successively riskier debt classes. Thus, we must limit the amount of debt that the firm can issue in each risk class to the amount acceptable to the

market. In practical terms we are placing limits on the amount of debt the firm can issue in the various bond rating categories, such as AAA, AA, etc.

The approach used to structure the debt limit constraints is to assume that the market will accept additional debt in a given risk class (quality rating) up to the point where the probability that the firm could "get into trouble" reaches an unacceptable level. We define trouble as a situation in which the real value of the firm's assets is less than the book value of its liabilities. Thus, the debt outstanding in each risk class must be limited to a fraction of the expected market value of total assets in each future period. The size of the fraction will depend on the degree of uncertainty about the future value of the firm's assets.

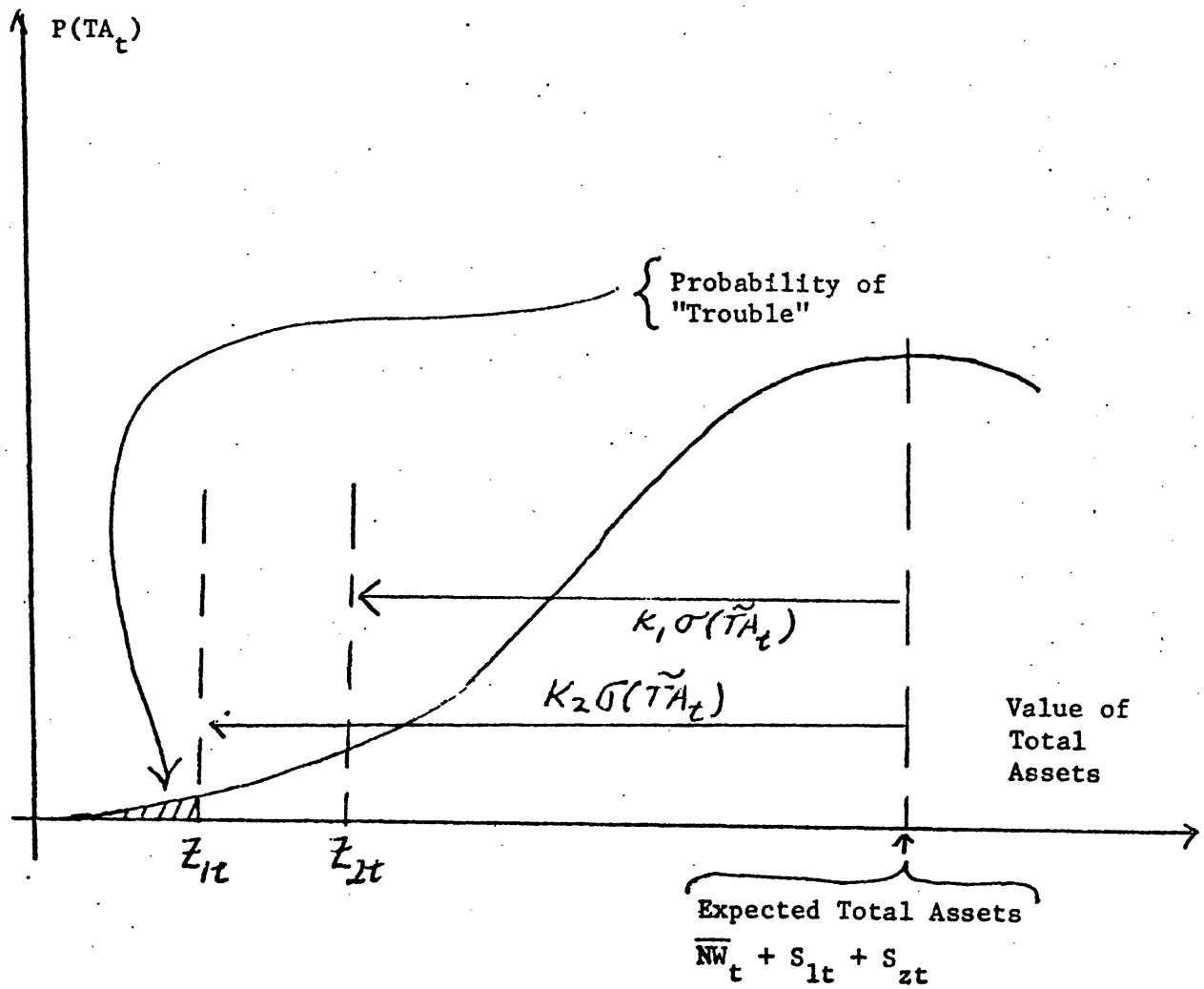
This is illustrated in Figure 3-1. The figure shows the distribution of the market value of the firm's total assets at the beginning of period t of the planning horizon. The Z_{1t} and Z_{2t} are the limits for class 1 debt and class 1 plus class 2 debt, respectively. Total assets equal the sum of the net worth (NW_t) plus the stock of class 1 and class 2 debt outstanding. In general, there would be i values of Z_{it} , one for each debt class.

The amount of risk class 1 debt outstanding in period t is obtained by summing the amounts outstanding from issues in periods 0 (initial debt) through t . That is,¹⁵

$$S_{1t} = \sum_{\tau=0}^t Y_{\tau 1} \cdot \beta_{\tau 1}^t \quad (3-2)$$

Figure 3-1

Debt Limit Based on the Uncertainty
About the Future Value of the Firm's Assets



The amount of debt outstanding in each risk class is limited to a portion of expected total assets. The appropriate fraction is obtained by restricting the amount of debt in the first i classes such that the probability of "trouble" is equal to the maximum allowable value. That is,

$$P(\tilde{TA}_t \leq Z_{it}) = \epsilon_i \quad (3-3)$$

where Z_{it} is the dollar limit on the first i classes of debt. In principle, the probability limit ϵ_i is determined from the behavior of the debt markets. We use ϵ_i and the statistical properties of \tilde{TA}_t to solve for the debt limit, Z_{ti} .

From Eq. (3-3),

$$E(\tilde{TA}_t) = Z_{ti} + k_i \sigma(\tilde{TA}_t) \quad (3-4)$$

where k_i is equal to $F^{-1}(\epsilon_i)$, F being the cumulative probability distribution for \tilde{TA}_t .¹⁶ Thus,

$$Z_{ti} = E(\tilde{TA}_t) - k_i \sigma(\tilde{TA}_t), \quad (3-5)$$

where

$$E(\tilde{TA}_t) = E[NW_t] + \sum_{i=1}^I S_{it} \quad (3-6)$$

and

$$\sigma^2(\tilde{TA}_t) = \sigma^2(NW_t) = \sigma^2(\tilde{A}_p^t).$$

where \tilde{A}_p^t is the present value of the firm's portfolio of investment projects in period t . Now

$$\sigma^2(\tilde{A}_p^t) = \sum_{k=0}^N \sum_{k'=0}^N X_k X_{k'} \text{Cov}(\tilde{A}_k^t, \tilde{A}_{k'}^t),$$

which is, of course, non-linear in the X_k 's. However,

$$\begin{aligned}\sigma^2(\tilde{A}_p^t) &= \sum_{k=0}^N X_k \text{Cov}(\tilde{A}_k^t, \tilde{A}_p^t) \\ &= \sum_{k=0}^N X_k \gamma_{kt} \sigma(\tilde{A}_k^t) \sigma(\tilde{A}_p^t)\end{aligned}$$

where γ_{kt} is the correlation coefficient between \tilde{A}_k^t and \tilde{A}_p^t .

Therefore,

$$\sigma(\tilde{A}_t) = \sigma(\tilde{A}_p^t) = \sum_{k=0}^N X_k \gamma_{kt} \sigma(\tilde{A}_k^t) \quad (3-7)$$

Eq. (3-7) does not really eliminate non-linearities in the specification of $\sigma^2(\tilde{A}_t)$, since γ_k depends on the projects accepted and thus is not available ex ante. However, it is a tolerable linear approximation if we can assume that the correlation between \tilde{A}_k^t and \tilde{A}_p^t is equal to that between \tilde{A}_k^t and \tilde{A}_0^t , the present value of the firm's assets existing at $t = 0$. This is realistic if the firm's investment opportunities do not call for ventures into entirely different industries, or if such ventures are small relative to the existing assets and other opportunities.

Substituting Eqs. (3-6) and (3-7) into (3-5) we have

$$Z_{ti} = E[NW_t] + \sum_{i=1}^I S_{ti} - k_i \sum_{k=0}^N X_k \gamma_k \sigma(\tilde{A}_k^t) \quad (3-8)$$

with $k_{i-1} > k_i$ for all i . The limit on class i debt can now be expressed

$$\text{as } S_{it} + W_{it} = Z_{it} - Z_{i-1,t}, \quad (3-9)$$

where W_{it} is the unused class i debt capacity in period t .

3. Liquidity Reserve

The sources and uses equations are stated in terms of expectations. However, given the uncertainties associated with project cash flows, planned sources may be insufficient to meet actual cash requirements in future time periods. For any real firm, both management and stockholders would wish to ensure that a sufficiently large liquidity buffer is built into the financial plan to provide a degree of flexibility in the face of uncertainties associated with future cash flows. The liquidity reserve (LR) is composed of liquid assets plus unused borrowing potential,

$$LR_t = L_t + \sum_{i=1}^I W_{it} \quad (3-10)$$

The various components which make up future cash requirements were enumerated in Eq. (3-1). For purposes of this model, we assume that uncertainty is confined to project cash flows, $\tilde{C}_k(t)$.

The constraint implied by the liquidity reserve requirement is given by

$$\text{PROB}(\tilde{C}R_t \leq CS_t + LR_t) \geq 1 - \epsilon_t \quad (3-11)$$

where $\tilde{C}R_t$ = the cumulative project cash requirements to period t (a random variable);

CS_t = the cumulative financing cash flows to period t (a deterministic variable), and

ϵ_t = the maximum acceptable probability of having insufficient cash available in periods 1 through t .

The constraint is formulated on a cumulative rather than a noncumulative basis so that the statistical relationship among the period-by-period

project cash flows can be explicitly considered. Also, the cumulative formulation avoids the problem of double-counting the same liquidity reserve against liquidity requirements in several periods.

From equation (3-11),

$$CS_t + LR_t - E(\tilde{CR}_t) \geq h_t \sigma(\tilde{CR}_t), \quad (3-12)$$

where h_t is equal to $F^{-1}(1-\epsilon_t)$, F being the cumulative probability distribution for \tilde{CR}_t . However, from equation (3-1) planned requirements $E(\tilde{CR}_t)$ are equal to planned sources CS_t ; thus, (3-12) becomes

$$LR_t \geq h_t \sigma(\tilde{CR}_t),$$

or

$$LR_t = L_t + \sum_{i=1}^I W_{it} \geq h_t \sigma(\tilde{CR}_t). \quad (3-13)$$

It now remains to develop an expression for $\sigma(\tilde{CR}_t)$ in terms of the uncertainties of the cumulative project cash flows. By definition

$$\tilde{CR}_t = \sum_{k=0}^N x_k \tilde{C}_{Ck}(t).$$

Therefore,

$$\sigma(\tilde{CR}_t) = \sum_{k=0}^N x_k \delta_{kt} \sigma[\tilde{C}_{Ck}(t)],$$

where $\tilde{C}_{Ck}(t)$ = the cumulative after tax cash flows for periods 1 through t for project k , and

δ_{kt} = the correlation coefficient between $\tilde{C}_{Ck}(t)$ and \tilde{CR}_t . We approximate γ_{kt} by the correlation between $\tilde{C}_{Ck}(t)$ and the autonomous cash flows $\tilde{C}_{C0}(t)$, using the same reasoning applied to γ_k in Eq. (3-7).

Finally, the expression for the liquidity reserve becomes

$$L_t + \sum_{i=1}^I W_{it} \geq h_t \sum_{k=0}^N X_k \delta_{kt} \sigma(\tilde{C}_{Ck}(t)) \quad (3-14)$$

Of course, Eq. (3-14) is in some respects similar to Eqs. (3-8), which limit planned debt levels. Both types of constraint protect the firm from overcommitment in the face of uncertainty. But the constraints are conceptually and practically different. The debt capacity constraints insure long-run solvency, but liquidity is by definition a short run concept. Thus, Eqs. (3-14) preserves flexibility rather than insolvency. The size of the liquidity reserve is critical only for the first few periods of the plan, in which management's ability to respond to unexpected events is limited. A reasonable means of reflecting this in the model is to let h_t approach zero as t approaches H , the horizon period.

Naturally, the definition of "short run" depends on the characteristics of the firm's business, in particular the speed and cost of revising investment plans and the predictability of future investment opportunities.¹⁷

4. Investment Project Constraints

The possibility that various interdependencies will exist among the investment projects requires the inclusion of additional constraints in the model. The most common interdependencies are mutual exclusion, contingency relationships, and physical dependencies between project cash flows.

If projects j and k are mutually exclusive, then the constraint

$$x_j + x_k \leq 1 \quad (3-15)$$

will ensure that only one project is accepted. (Remember that x_j and x_k are constrained to integer values.) This type of constraint can also be used to consider the possibility of accepting the same project now or at some later time.

If project j can be undertaken only if project k is accepted, the constraint is

$$x_j \leq x_k. \quad (3-16)$$

Physical dependencies between project cash flows occur when the cash flows associated with project j depend on whether project k is accepted, or vice versa. Pairwise physical dependencies can be handled by introducing a dummy project, w , which is included in the solution if and only if both projects j and k are accepted. Thus $X_w = 1$ whenever $X_j = X_k = 1$, and 0 otherwise.

Let A_w^1 represent the incremental present value obtained if both projects j and k are accepted. A_w^1 may be positive or negative.

When A_w^1 is positive, the model's "instinct" will always be to include project w . It is constrained from doing so by making the acceptance of project w contingent on the acceptance of both j and k . That is,

$$X_w \leq X_k \text{ and } X_w \leq X_j \quad (3-17)$$

When A_w^1 is negative, the following constraint is required:

$$X_w \geq X_j + X_k - 1 \quad (3-18)$$

That is, whenever $X_j = X_k = 1$, X_w must also equal 1.

Generalization of the treatment of physical dependencies to higher order relationships is straightforward. Suppose project w represents the effect of joint acceptance of M projects. If A_w^1 is positive, then

$$X_w \leq X_j, \quad j = 1, \dots, M \quad (3-19)$$

If A_w^1 is negative, then

$$X_w \geq \sum_{j=1}^M X_j - (M + 1). \quad (3-20)$$

An additional cause of project interdependencies results when special resources used by several projects are rationed. The firm may have a limited supply of managers to assign to the various projects under consideration. Foreexample, let

d_j = the amount of the scarce resource required by project j (e.g., the number of managers required)

D = the total supply of the resource available.

Then if m projects compete for the scarce resource, total usage must be limited to the amount D .

$$\sum_{j=1}^m d_j X_j \leq D. \quad (3-21)$$

cash flows.

5. Equity Issue Costs Constraints

Equity issue costs are composed of a fixed component F_t^e and a variable component v_t^e . For simplicity the variable cost rate is assumed to be independent of the issue size. The issue cost resulting from E_t dollars of equity issued in period t is

$$Z_t^e F_t^e + E_t v_t^e,$$

where Z_t^e is equal to zero if E_t equals zero, and one otherwise. To insure this relationship we required

$$E_t \leq Z_t^e \cdot \Omega \quad (3-22)$$

where Ω is a large number (e.g. 10^{10}). Thus, when Z_t^e is equal to zero, $E_t = 0$, and when $Z_t^e = 1$ the values of E_t are not limited by the constraint.

The market value of assets in period t , NW_t , will include the present value of (at t) all future equity issue costs. The expression for NW_t will thus include the terms

$$\sum_{\tau=t}^H [Z_{\tau}^e F_{\tau}^e + v_{\tau}^e E_{\tau}]$$

6. Informational Constraints

These constraints are used in the calculation of penalties to be applied to the firm's net worth resulting from erratic dividend and reported earnings policies. The effect of the constraints is to induce the model to smooth dividend payments and reported earnings over time. The formulation of the constraints for dividends and reported earnings are similar. Thus, we shall not provide a detailed treatment of both. Let D_t = the aggregate dividends paid in year t ($t = 0, 1, \dots, H$)

g_d = the expected long-run growth rate of dividends.

The difference between the dividends paid and the target level is equal to $(1 + g_d) D_{t-1} - D_t$. The impact on the market price of the firm's stock will increase with the magnitude of the reduction. Thus, the difference can be divided into steps which will be penalized at successively higher rates.¹⁸

We begin by dividing any decrease in aggregated dividend payments into increments, Δ_{tj}^d .

$$(1 + g_d) D_{t-1} - D_t \leq \sum_{j=1}^P \Delta_{tj}^d \quad (3-23)$$

$$t = 1, \dots, H$$

where Δ_{tj}^d is the amount of dividend reduction in the j^{th} penalty class ($j = 1, \dots, P$). Note that if $D_t \geq (1 + g_d) D_{t-1}$ then Δ_{tj}^d will equal zero for all values of j . The model assumes that dividend increases beyond the target level are neither rewarded nor penalized.

The amount of dividend reduction in each penalty class is limited to a specific fraction of the base dividend level. The first penalty class is for dividend increases less than the expected value, the rest for dividend reductions. Thus, $\Delta_{t1}^d \leq g_d D_{t-1}$, and for $j = 2, \dots, P$,

$$\Delta_{tj}^d \leq \theta_j D_{t-1} \quad (3-24)$$

where $\theta_j > 0$ and $\sum \theta_j = 1$. The concave shape of the penalty function ensures that $\Delta_{t,j-1}^d$ reaches its upper bound before Δ_{tj}^d takes on positive values. NW_t will include the term $\sum_{\tau=t}^H \sum_{j=1}^P \Delta_{tj}^d \cdot p_{tj}^{dc}(\tau)$ to reflect the market value penalty associated with dividend reductions.

The penalty costs associated with reductions in reported earnings are treated in the same manner. Reported earnings in period t are given by

$$RE_t = \sum_{k=0}^N X_k \cdot RE_{kt} + L_{t-1} \cdot d_L(t) - \sum_{\tau=0}^t \sum_{i=1}^I Y_{\tau i} d_{\tau i}(t) \quad (3-25)$$

where RE_{kt} = contribution to reported earnings of project k in year t ;

$d_L(t)$ = after-tax interest rate received in period t per dollar of liquid assets held during period $t-1$, and

$d_{\tau i}(t)$ = after-tax interest payments in period t per dollar of risk class i debt issued in period τ .

The target reported earnings in period t are equal to $(1 + g_R)RE_{t-1}$.

Any decrease in expected earnings from the target level can be divided into segments Δ_{tj}^{re} which are penalized at successively higher rates.

$$(1 + g_R)RE_{t-1} - RE_t \leq \sum_{j=1}^Q \Delta_{tj}^{re} \quad (3-26)$$

Where, as in the dividend case,

$$\Delta_{t1}^r \leq g_r^{RE}{}_{t-1} \quad (3-27)$$

$$\Delta_{tj}^r \leq \alpha_j^{RE}{}_{t-1} \quad j=2, \dots, Q,$$

$$\alpha_j > 0.$$

The net worth in period t will be reduced by an amount

$$\sum_{\tau=t}^H \sum_{j=1}^Q \Delta_{tj}^r P_{tj}^{re}(\tau)$$

reflecting the reporting earnings reduction penalties.

Model Summary

We can now present a complete algebraic expression for the net worth in period t.

$NW_t = \sum_{k=0}^N X_k A_k^t$	Present value in period t of project and autonomous cash flows
$+ \sum_{\tau=t}^H L_{\tau} A_{\tau L}^t$	Present value of liquid assets held during periods t through H
$+ \sum_{\tau=0}^H \sum_{i=1}^I Y_{\tau i} F_{\tau i}^t$	Present value of financing options
$+ \sum_{\tau=t}^H \sum_{i=1}^I S_{i\tau} P_{\tau i}^b(\tau)$	Present value of bankruptcy costs
$+ \sum_{\tau=t}^H [Z_{\tau}^e F_{\tau}^e(\tau) + E_{\tau}^e v_{\tau}^e(\tau)]$	Present value of equity issue costs
$+ \sum_{\tau=t}^H D_{\tau} P_{\tau}^t(\tau)$	Present value of dividend tax penalty

$$+ \sum_{\tau=t}^H \sum_{j=1}^P \Delta_{\tau j}^d P_{tj}^{dc}(\tau) \quad \text{Present value of dividend reduction penalties}$$

$$+ \sum_{\tau=t}^H \sum_{j=1}^Q \Delta_{\tau j}^r P_{tj}^{re}(\tau) \quad \text{Present value of reported earnings reduction penalties}$$

The objective of the model is to maximize the net worth at the beginning of period 1, NW_1 . The major constraints on the decision variables are,

(a) Sources and Uses of Cash

$$\begin{aligned} & \sum_{k=0}^N X_k C_k(t) - \sum_{\tau=0}^t \sum_{i=1}^I Y_{\tau i} f_i(t) \\ & \{ + E_t[1 - v_t^e(t)] - Z_t^e F_t^e(t) \} - D_t \\ & - \{ L_t - (1 + (1 - T_c) \rho_L) L_{t-1} \} \end{aligned} \quad (3.1)$$

(b) Debt Capacity

- Stock of class i debt in period t

$$S_{it} = \sum_{\tau=0}^t Y_{\tau i} \beta_{\tau i}^t \quad (3.2)$$

- Limit for class i debt

$$S_{it} + W_{it} = Z_{it} - Z_{i-1,t} \quad (3.9)$$

for all values of i.

(c) Liquidity Reserve

$$L_t + \sum_{i=1}^I W_{it} \geq h_t \sum_{k=0}^N X_k \delta'_{kt} \sigma(\tilde{C}_{ck}(t)) \quad (3.13)$$

(d) Project Constraints

- positive interaction ($A_w^1 > 0$)

$$X_w \leq X_j \quad j = 1, \dots, M \quad (3.19)$$

- negative interaction ($A_w^1 < 0$)

$$X_w \geq \sum_{j=1}^M X_j - (M + 1) \quad (3.20)$$

(e) Equity Issue Costs

$$E_t \leq Z_t^e \cdot \Omega \quad (3.22)$$

(f) Information Effects

- dividend cuts

$$(1 + g_d) D_{t-1} - D_t \leq \sum_{j=1}^P \Delta_{tj}^d \quad (3.23)$$

$$\text{where } \Delta_{tj}^d \leq \theta_j \cdot D_{j-1} \quad (3.24)$$

- reported earnings reductions

$$(1 + g_r) RE_{t-1} - RE_t \leq \sum_{j=1}^Q \Delta_{tj}^r \quad (3.26)$$

$$\text{where } \Delta_{tj}^r \leq \alpha_j \cdot RE_{t-1} \quad (3.27)$$

Each of the above constraints must hold for values of t from 1 to the horizon period H .

IV. INVESTMENT OPPORTUNITIES AND HORIZON CONDITIONS

Two important aspects of the model remain to be discussed -- the completeness of the investment opportunity set and the conditions at the end of the planning horizon.

Investment opportunity set

Thus far we have implicitly assumed that all investment projects to be considered during the planning horizon can be identified in project by project detail at the beginning of the planning period. As a practical matter, this is impossible. While the financial manager may be able to describe all potential investment projects in the early years of the planning horizon, this will not be the case for the later periods. At most, he will be able to describe the aggregate nature of yet-to-emerge opportunities.

In the absence of interdependencies among various investment and financing decisions, there would be no need for additional information about these opportunities; however, the sort of interdependencies discussed in this paper makes it impossible to ignore them, since the likelihood of future opportunities will influence current investment and financing decisions.

A practical solution is to define a series of yet-to-emerge investment projects, one for each year of the planning horizon, from $t=2$ to $t=H$. These would represent the financial managers' best guesses as to the new opportunities that will eventually arise for those years. The magnitude, risk and duration of the cash flows would probably reflect extrapolations of previous investment experience. The amount of potential investment opportunities would likely increase for

later periods in the planning horizon, complementing the managers' declining detailed knowledge of specific projects. The result of this approach is to permit the delay of commitments to specific future projects until later when more information becomes available.

To include these projects in the model, we define a series of decision variables X_{N+t-1} , for $t=2, \dots, H$. These variables are continuous rather than discrete, allowing fractions of the total of projected opportunities to be included in the financial plan. Otherwise, these projects would be treated identically to the N identified projects.

By including these projects we have prevented the apparent "disappearance" of the firm in later periods due to the absence of specific projects. Without them the first period decisions would be biased by the apparent short life of the firm. For example, if future opportunities were ignored, the debt capacity of the firm would appear to decline and the model would be forced to plan for early retirement of debt.

Horizon Conditions

Another set of problems arise from the fact that the model looks ahead only a finite number of periods, whereas the firm will continue to exist well beyond that time. It is thus necessary to consider how the myopic nature of the model effects the recommended financial plan. That is, what biases will result in the plan from the finite nature of the model, and how can they be corrected? Ideally, we would like to interface the model with the periods beyond the horizon so that decisions are made as if the planning horizon were infinite.

One problem arises when the model is extended to cover the choice among debt maturities. In this case there will be issues with repayment schedules extending beyond the horizon. Since longer maturity debt options have higher net present values per dollar issued, and since there are no debt capacity constraints beyond $t=H$, the model will use maximum maturity options for financing extending beyond the horizon. Thus, the horizon debt structure will be artificially biased toward longer maturities.

To remove this bias we truncate all debt issued at the beginning of period $H+1$. Thus, all debt extending beyond the horizon is assumed to mature at the beginning of period $H+1$, even though the interest rates are for longer maturity periods. With this change the debt option net present values F_{Tij}^t (j is the coefficient for maturity and type) will no longer bias the model toward longer maturity instruments.

Truncation of debt at the planning horizon, however, will result in a bias against longer-lived projects. This results from eliminating the benefits of post horizon debt capacity, which will be largest for longer lived projects.

The solution clearly requires an approximation for the value of the post-horizon debt capacity related to each project. One way to do this is to discount project k 's post horizon expected cash flows back to $t=1$ at a weighted average cost of capital ρ_k^* , while continuing to discount pre-horizon flows at ρ_k , the appropriate rate assuming all-equity financing. Specifically, we replace Eq. (2-2) with:

$$A_k^1 = \sum_{\tau=1}^H \frac{C(\tau)}{(1+\rho_k)^{\tau-1}} + \sum_{\tau=H+1}^{\infty} \frac{C(\tau)}{(1+\rho_k^*)^{\tau-1}} \quad (4.1)$$

where ρ_k^* is the weighted average cost of capital for the project. We use the MM formula^(from [15]) to obtain ρ_k^* from ρ_k and λ_k , a target debt ratio for the project:

$$\rho_k^* = \rho_k(1 - T_c \lambda_k). \quad (4.2)$$

The effect of discounting ρ_k^* rather than ρ_k is to increase the present value of post horizon cash flows (assuming λ_k positive), thereby reflecting the project's contribution to post-horizon debt capacity. This method is not exact, but the errors introduced should not be serious in the present context.¹⁹

Beyond period H, and after the lifetimes of the yet-to-emerge projects discussed above, we assume that the firm enters a steady state. In this condition, it is assumed to invest in projects which make no contribution to net worth. While it would be conceptually possible to specify the nature of post-horizon growth opportunities, in practice this would be a nebulous affair. Also, given their remoteness, the effect on current decisions will tend to be small.

V. CONCLUDING REMARKS

Our purpose was to develop a model which would permit simultaneous consideration of the investment, financing and dividend options facing the firm. Financial theory has typically treated these as independent decisions. However, this has been at the cost of ignoring a number of important interactions. The main considerations which lead to a requirement for simultaneous solution are:

- (1) Corporate Debt Capacity, which depends on the total risk characteristics of the asset portfolio and not on individual projects per se.
- (2) Liquidity requirements, based on the uncertainty of aggregate portfolio cash flows.
- (3) Fixed costs associated with equity issues.
- (4) Project interdependencies and resource constraints.
- (5) Informational problems associated with dividend and reported earnings policies.

These considerations are present in most practical situations. Given this, it is not possible to maximize the value of the firm through simple sequential determination of investment, financing and dividend decisions. Our approach has incorporated these interactions and uses mathematical programming to obtain the jointly optimal set of decisions.

The approach has a number of distinct advantages in addition to the simultaneous nature of the solution. The major ones are:

- (1) There are no restrictions on project risk characteristics. The model allows explicit treatment of project risk differences in both the objective function and constraints.
- (2) Debt capacity is based on project risk characteristics and not on arbitrarily determined debt equity ratios.
- (3) The liquidity reserve is similarly based on the risk of project cash flow, and not on rules of thumb.
- (4) The model uses mixed integer linear programming in order to avoid the fractional project difficulties associated

with ordinary linear programming. Computationally efficient codes exist for practical size problems.

The model formulation has a number of minor problems and one more significant difficulty.

- (1) Minor problems result from the assumptions necessary to maintain the linear structure of the debt capacity and liquidity reserve constraints. Also, penalties associated with dividend and reported earnings reductions are on an aggregate rather than per share basis to avoid the non-linearities associated with per share calculations. The latter is only a problem during periods when new equity is issued.
- (2) The more serious problem results from the non-sequential nature of the model. The model treats the problem as a single rather than multi-stage decision problem. That is, the model develops a complete financial plan at the beginning of the planning horizon rather than a set of decision rules which will guide the course of the plan as new information becomes available. Thus, decisions for periods 2 through H do not reflect the opportunity to obtain new information at the end of period 1, and hence will no longer be optimal when new information is received at the end of period 1. Of course, there is no need actually to implement these decisions, since the model can be re-run at the beginning of each year to produce an updated financial plan.

FOOTNOTES

1. See Myers [17] for a detailed treatment of the weighted average cost of capital and the errors that can result from its use.
2. There are some minor exceptions (noted in Section III below) where approximations must be made to maintain the linear structure.
3. In addition to specific references later in the paper, we must note here our general reliance on several authors' work.

The basic linear programming framework for financial planning under certainty was laid by Charnes, Cooper and Miller [3]. Weingartner [24] extended this approach to capital budgeting decisions, and N  slund [18] made progress in extending the Weingartner model to conditions of certainty.

The financing side of the model relies heavily on Modigliani and Miller's treatment of capital structure and dividend policy. [11,12,13]

Although the analysis of the risk of investment projects is exogenous to the model, the model does rely on the notion of risk independence introduced by Myers [15]. As Fama [4] and Hamada note [6], risk independence also holds approximately in the capital asset pricing model of Sharpe [23], Lintner [10] and Mossin [14].

Of course, other optimization models have been proposed for financial planning, but the ones we have seen (examples are Carleton [2], Chambers [3a] and Moses [5]) do not share the essential features of the model we are presenting here. By "essential features" we have in mind particularly our reliance on modern capital market theory, our treatment of risk differences among investment opportunities, and our specification of debt capacity and liquidity constraints.

4. That is, it is assumed that the average interest rate with subordination is the same, *ceteris paribus*, as if one undifferentiated class of debt were issued. This seems to be a reasonable approximation given the intent of the model. See Robichek and Myers [21], pp. 31-32 for a theoretical justification.

5. See Myers [15], and Schall [22]. The latter provides a review of the literature on this subject.

The risk-independence concept, usually advanced with the firm's investment decision in mind, applies to the financing decision as well. (That is, the proofs apply to transactions either in real or financial assets.) Thus, this concept also supports our use of a linear objective function with respect to security issues and retirements.

6. See Modigliani and Miller [13], and Robichek and Myers [21].
7. For ease of exposition this one period debt assumption will be maintained throughout Section II. In later sections the discussion will be generalized to debt of fixed, multi-period maturity.
8. There are no conceptual difficulties in expanding Eq. (2-4) to cover various maturities, changing interest rates across maturities and time, etc.
9. Of course, this assumes that T_c is known, that the firm will have taxable income in all future periods, etc. It also assumes the firm always borrows at the going rate ρ_1 . If this was not true for some reason (example: government-subsidized financing) then the present value of the debt option would be computed from Eq. (2-4) rather than (2-5).
10. For discussion, see Jaffee and Modigliani [7] and the other sources cited there.
11. Note that the bankruptcy costs are not due to imperfections in the (secondary) securities markets. Rather, they are real costs (e.g., lawyer's fees) deducted from the firm's assets in the event of bankruptcy. Thus, the introduction of bankruptcy costs here does not conflict with our reliance on perfect capital markets, the idea of risk independence or with MM's basic approach to the analysis of financing decisions.

12. Miller and Modigliani [11], esp. pp. 412, 415. The proof was independently presented by Lintner [9]; however, Lintner feels its applicability is much more limited than MM do.
13. These costs may also reflect the clientele effect stressed by Lintner [9].
14. See Black and Scholes [1] for recent tests and references to earlier work.
15. For the one period debt case the stock of class i debt outstanding is given by

$$S_{it} = \beta_{oi}^t Y_{oi} + Y_{ti}.$$

16. k_i can be determined by assuming \tilde{TA}_t is normally distributed, or from a nonparametric relationship such as Tchebyscheff's inequality.
17. The need for the explicit inclusions of a liquidity reserve constraint results from the single-stage nature of the model. Specifically, the sources and uses constraints for $t=1, \dots, H$ are based on expected (as of $t=0$) cash requirements. The liquidity reserve is required to insure that sufficient cash could be available to meet the actual requirements in period 2 and beyond. Without this reserve, additional costs (not considered in the current objective function) would be incurred in raising additional cash to meet the unanticipated requirements. If the model had been formulated as a multi-stage decision problem, explicit recognition would have been taken of the requirements that could arise beyond the first period. The optimal financial plan would then have build in liquidity reserves to protect against higher than expected cash requirements. However, for practical reasons, as well as ease of exposition we have chosen not to follow the multi-stage approach. The data and computational requirements for a full blown multi-stage approach are typically prohibitive for problems of practical size. From an exposition point of view formulating

the model as a multi-stage linear programming problem would simply add detail which would obscure discussion of the basic issues. This approach is identical to that used by Pogue and Bussard in their short term planning model [19].

18. Another approach to smoothing dividends and reported earnings is commonly used in analytical planning models (see, for example, Lerner and Rappaport [8]). It involves simply constraining aggregate values in each period to equal or exceed one plus the target growth rate times the value in the previous period. The costs of these constraints are then evaluated through an examination of the relevant shadow prices.

Our approach allows the model to "price out" reductions in dividends and reported earnings. However, the simpler approach may be more appealing in some applications.

19. See Myers [17], pp. 25-32.

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